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Algebras of Cauchy Continuous Maps

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A function f between two uniform spaces X and Y is Cauchy continuous if it maps every Cauchy filter on X to a Cauchy filter on Y. In the context of uniform spaces these functions are studied in [12] where they are called "Cauchy regular functions." The collection of Cauchy continuous functions has much nicer algebraic properties than the collection of uniformly continuous functions. On the other hand Cauchy continuous functions share the fundamental extension property of uniformly continuous maps. In fact the collection of Cauchy continuous maps from a uniform space X to a complete uniform space Y is exactly the class of functions with continuous extensions to the completion of X.

But the study of Cauchy continuous maps really belongs to the field of Cauchy spaces. In fact Cauchy continuous maps are the morphisms in the category of Cauchy spaces, a category which has been created to develop a general completion theory [7, 10, 11].

In this paper we will be concerned with real-valued Cauchy continuous maps on a Cauchy space $(X, \mathscr{C})$. We will study the properties of the collection $\Gamma(\mathscr{C})$ of all real-valued Cauchy continuous maps rather than the properties of the maps themselves. The question that will be answered in this paper is: What properties characterize the collections $\Gamma(\mathscr{C})$ among all function classes on X (i.e., all subsets of $\mathbb{R}^X$)? For any function class Φ on X, the filters on Φ "mapping" Φ-filters on X to convergent filters on $\mathbb{R}$ are natural Cauchy filters. The collection γ_Φ of all these Cauchy filters is a Cauchy structure on Φ. It plays the key role in the characterization of the classes $\Gamma(\mathscr{C})$. The main theorem states that the classes $\Phi = \Gamma(\mathscr{C})$ are exactly the γ_Φ-complete unitary function algebras. For classes of bounded functions γ_Φ-completeness may be replaced by uniform completeness. Combining these results with the characterization obtained earlier in [9], that the collections $\Gamma(\mathscr{C})$ are exactly the composition closed function classes, we answer problem (1) in [4].

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1. Preliminaries

For the theory of Cauchy spaces and Cauchy continuous maps we refer to [7, 11 and 12].

Throughout the paper we use the following notations. X is a set and $\mathscr{C}$ is a *Cauchy structure* on X, i.e., a collection of filters on X satisfying the conditions that for every point x of X the filter $\dot{x}$ generated by $\{x\}$ belongs to $\mathscr{C}$; if a filter belongs to $\mathscr{C}$ then so does every finer filter and if two filters in $\mathscr{C}$ have a supremum then their intersection belongs to $\mathscr{C}$. The filters belonging to $\mathscr{C}$ are the *Cauchy filters*. $\Gamma(\mathscr{C})$ is the collection of all real-valued *Cauchy continuous* maps, i.e., the functions that map Cauchy filters on X to convergent filters on $\mathbb{R}$. We call $\mathscr{C}$ *point separated* if $\Gamma(\mathscr{C})$ separates the points of X. A Cauchy structure is *uniformizable* if it is the collection of Cauchy filters in some Hausdorff uniformity. Every uniformizable Cauchy structure is point separated. If $\mathscr{U}$ is a uniformity on X, the collection of *uniformly continuous* real-valued maps is denoted $U(\mathscr{U})$ and $U(\mathscr{U}) \subset \Gamma(\mathscr{C})$ if $\mathscr{C}$ is the collection of $\mathscr{U}$-Cauchy filters. A Cauchy structure is *totally bounded* if it contains all ultrafilters.

For any Cauchy structure $\mathscr{C}$ on X we use the *associated convergence structure* q to describe continuity, Hausdorffness, closure, density, etc. A filter $\mathscr{F}$ q-converges to x if and only if $\mathscr{F} \cap \dot{x}$ belongs to $\mathscr{C}$. The structures $\mathscr{C}$ and q are called *compatible* if they are related in this way. If $\mathscr{C}$ is point separated then q is Hausdorff, and $\mathscr{C}$ is *complete* if every Cauchy filter q-converges. $\mathfrak{C}(q)$ $[\mathfrak{C}^*(q)]$ is the collection of real-valued [bounded] *continuous maps* on (X, q). The inclusion $\Gamma(\mathscr{C}) \subset \mathfrak{C}(q)$ always holds.

For a survey of the theory of function classes we refer to [3] and [4]. We recall some of the definitions. A *function class* on X is a subset of $\mathbb{R}^X$. The notions function algebra or function lattice are defined by means of the pointwise operations. A function class Φ is *point separating* if it separates the points of X. Φ is *composition closed* if the following is true: given $\{f_i \mid i \in I\} \subset \Phi$ and $h: X \to \mathbb{R}^I$, $h(x) = (f_i(x))_{i \in I}$ for $x \in X$, the map $k \circ h$ belongs to Φ whenever k is a continuous real-valued map on the closure of $h(X)$ in $\mathbb{R}^I$. A filter $\mathscr{F}$ on X is a *Φ-filter* if and only if $f(\mathscr{F})$ converges whenever f belongs to Φ. The collection $c(\Phi)$ of all Φ-filters is the coarsest Cauchy structure on X making all maps in Φ Cauchy continuous, and it is called the *weak Cauchy structure of Φ*. If Φ is point separating then $c(\Phi)$ is point separated. The *weak topology of Φ* is denoted by $t(\Phi)$ and it is compatible with $c(\Phi)$. *The weak uniformity* of Φ is denoted by $u(\Phi)$. The Cauchy structure $c(\Phi)$ is uniformizable, it is the collection of all $u(\Phi)$-Cauchy filters.

From now on by "Cauchy structure $\mathscr{C}$ on X" we mean "point separated Cauchy structure" and by "function class Φ" we mean a "point separating function class" on X. These assumptions are made throughout the paper.

2. Duality between Cauchy Structures and Function Classes

In this section we expand some results of [8] and [9]. Starting with a Cauchy structure $\mathscr{C}$ on X, associating $\Gamma(\mathscr{C})$ and then taking the weak Cauchy structure of $\Gamma(\mathscr{C})$ one obtains a Cauchy structure $c(\Gamma(\mathscr{C}))$ which is coarser than $\mathscr{C}$ and is uniformizable. The structures $\mathscr{C}$ and $c(\Gamma(\mathscr{C}))$ have the same collections of Cauchy continuous maps, i.e., $\Gamma(\mathscr{C}) = \Gamma(c(\Gamma(\mathscr{C})))$. We call $c(\Gamma(\mathscr{C}))$ the *uniformizable modification* of $\mathscr{C}$ and denote it by $\mathscr{C}_u$. For two Cauchy structures $\mathscr{C}$ and $\mathscr{D}$ on X the equality $\Gamma(\mathscr{C}) = \Gamma(\mathscr{D})$ holds if and only if $\mathscr{C}_u = \mathscr{D}_u$.

Assuming non-measurable cardinality for X then $\mathscr{C}_u$ is the finest uniformizable Cauchy structure on X coarser than $\mathscr{C}$ and $\mathscr{C}_u$ is a reflection of $\mathscr{C}$ in the category of uniformizable Cauchy spaces. Again assuming non-measurable cardinality $c\Gamma(\mathscr{C}) = \mathscr{C}$ holds if and only if $\mathscr{C}$ is uniformizable.

Starting with a function class Φ on X, associating $c(\Phi)$ and then taking $\Gamma(c(\Phi))$ one obtains a composition closed function class containing Φ. The classes Φ and $\Gamma(c(\Phi))$ have the same weak Cauchy structure, i.e., $c(\Phi) = c(\Gamma(c(\Phi)))$. We call $\Gamma(c(\Phi))$ the *composition closed modification* of Φ and denote it Φ_c. Two function classes have the same weak Cauchy structure if and only if they have the same composition closed modification. Φ_c is the smallest composition closed function class containing Φ and $\Gamma(c(\Phi)) = \Phi$ holds if and only if Φ is composition closed.

From the previous results we can conclude that there exists a one to one correspondence between the collection of uniformizable Cauchy structures on X and the collection of composition closed function classes on X.

3. The Cauchy Continuous Structure on a Function Class

If Φ is a function class on X, $\mathscr{A} \subset \Phi$ and $F \subset X$ let $\mathscr{A}(F) = \{f(x) \mid f \in \mathscr{A}, x \in F\}$. If ψ is a filter on Φ and $\mathscr{F}$ is a filter on X let $\psi(\mathscr{F})$ be the filter on $\mathbb{R}$ generated by the collection $\{\mathscr{A}(F) \mid \mathscr{A} \in \psi, F \in \mathscr{F}\}$.

Definition 3.1. A filter ψ on Φ belongs to γ_Φ if and only if $\psi(\mathscr{F})$ converges whenever $\mathscr{F}$ is a Φ-filter on X.

Theorem 3.1. *γ_Φ is a point separated Cauchy structure on Φ.*

Proof. To prove that γ_Φ is a Cauchy structure is an easy verification. If f and g are different functions in Φ and $x \in X$ is such that $f(x) \neq g(x)$ then the map α_x from Φ to $\mathbb{R}$ which maps h to $h(x)$ is Cauchy continuous on (Φ, γ_Φ) and separates f and g.

We call γ_Φ *the Cauchy continuous structure* on Φ.

It is compatible with the algebraic and lattice operations as follows from the following results.

THEOREM 3.2. *If Φ is a function algebra (a function lattice) and carries γ_Φ, the algebraic (lattice) operations are all Cauchy continuous.*

Proof. We give a proof for the addition $s: \Phi \times \Phi \to \Phi$, $s(f, g) = f + g$. The other proofs are similar. Let $\psi \in \gamma_\Phi$ and $\psi' \in \gamma_\Phi$ and $\mathscr{F}$ a Φ-filter on X. Then $\psi(\mathscr{F}) + \psi(\mathscr{F}')$ converges and $s(\psi \times \psi')(\mathscr{F}) \supset \psi(\mathscr{F}) + \psi(\mathscr{F}')$. So $s(\psi \times \psi')$ belongs to γ_Φ. ∎

COROLLARY 3.3. *If Φ is a function algebra (function lattice) and $\Phi' \subset \Phi$ is a subalgebra (sublattice) then so is the closure of Φ' in (Φ, γ_Φ).*

On some special function classes other natural structures were introduced earlier. On a function class $\Phi = \Gamma(\mathscr{C})$ a natural Cauchy structure $\hat{\mathscr{C}}$ was introduced in [5] in the following way: ψ belongs to $\hat{\mathscr{C}}$ if and only if $\psi(\mathscr{F})$ converges whenever $\mathscr{F}$ belongs to $\mathscr{C}$.

THEOREM 3.4. *If X has non-measurable cardinality, $\mathscr{C}$ is a uniformizable Cauchy structure on X and $\Phi = \Gamma(\mathscr{C})$ then γ_Φ coincides with $\hat{\mathscr{C}}$.*

Proof. Under the hypothesis of the theorem $\mathscr{C} = c(\Gamma(\mathscr{C}))$ holds. So $\mathscr{C}$ is exactly the collection of Φ-filters. ∎

Remarks 3.5. 1. In general on $\Phi = \Gamma(\mathscr{C})$ the structures γ_Φ and $\hat{\mathscr{C}}$ are different. If $\mathscr{C}$ is $c^\wedge$ embedded [5] and $\hat{\mathscr{C}} = \gamma_\Phi$ then $(X, \mathscr{C})$ and $(X, c(\Gamma(\mathscr{C})))$ have the same natural completions in the sense of [5]. So $\mathscr{C}$ and $c(\Gamma(\mathscr{C}))$ coincide.

2. When $\mathscr{C}$ is complete and uniformizable and X has non-measurable cardinality then γ_Φ is the collection of all continuously convergent filters. So it is clear from the results of [2] that γ_Φ is in general not a uniformizable, not even a topological, structure on Φ.

For any function class Φ we have $\Phi \subset \Gamma(c(\Phi)) \subset \mathfrak{C}(t(\Phi))$. So Φ inherits Cauchy structures from $\Gamma(c(\Phi))$ and $\mathfrak{C}(t(\Phi))$: $\Gamma(c(\Phi))$ has the structure $\widehat{c(\Phi)}$ from [5] defined above. On $\mathfrak{C}(t(\Phi))$ we consider the uniformity of uniform convergence [13] and also the uniform convergence structure $\mathscr{G}_c$ of continuous convergence defined in [2].

We will compare the Cauchy structures induced on Φ with γ_Φ.

THEOREM 3.6. *γ_Φ coincides with the structure induced by $\widehat{c(\Phi)}$.*

Proof. The collection of Φ-filters is exactly $c(\Phi)$. ∎

THEOREM 3.7. *γ_Φ is coarser than the Cauchy structure of uniform convergence.*

Proof. If ψ is a Cauchy filter for the structure of uniform convergence, $\mathscr{F}$ is a Φ-filter on X and V is an entourage on $\mathbb{R}$, choose an entourage U such that $U^3 \subset V$. Let $\mathscr{A} \in \psi$ such that for every f and g in $\mathscr{A}$ we have $(f(x), g(x)) \in U$ for every $x \in X$. Fix $h \in \mathscr{A}$ and choose $F \in \mathscr{F}$ such that $h(F) \times h(F) \subset U$. Then $\mathscr{A}(F) \in \psi(\mathscr{F})$ and $\mathscr{A}(F) \times \mathscr{A}(F) \subset V$. ∎

Remark 3.8. In general γ_Φ is strictly coarser than the Cauchy structure of uniform convergence. For example, take $X = \mathbb{R}$ with the usual complete Cauchy structure and Φ the collection of (Cauchy) continuous maps. The sequence $f_n(x) = (1/n)x$, $n \geqslant 1$, generates a filter ψ on Φ belonging to γ_Φ. However, ψ is not a Cauchy filter for the uniformity of uniform convergence.

On function classes of bounded functions, however, the two structures coincide. This will be discussed in Section 5.

THEOREM 3.9. *γ_Φ is finer than the Cauchy structure induced on Φ by the collection of Cauchy filters of the uniform convergence structure of continuous convergence.*

Proof. If ψ is γ_Φ-Cauchy and $\mathscr{F}$ $t(\Phi)$-converges to x then $\mathscr{F}$ is a Φ-filter and $\psi(\mathscr{F})$ converges. It follows that $\psi \times \psi$ belongs to $\mathscr{G}_c$. ∎

Remark 3.10. In general, γ_Φ is strictly finer than the $\mathscr{G}_c$-Cauchy structure. For example, take $X = \,]0, 1]$ with the usual Cauchy structure $\mathscr{C}$ and $\Phi = \Gamma(\mathscr{C})$. The sequence $(f_n(x))_{n \geqslant 1}$ with $f_n(x) = 0$ for $x \geqslant 1/n$ and $f_n(x) = n - n^2 x$ for $x < 1/n$ generates a filter ψ on Φ which continuously converges on $\mathfrak{C}(t(\Phi))$. So ψ is $\mathscr{G}_c$-Cauchy. On the other hand the trace $\mathscr{M}$ of the usual neighborhoodfilter of 0 on $]0, 1]$ is a Φ-filter and $\psi(\mathscr{M})$ does not converge. So ψ does not belong to γ_Φ.

4. COMPLETE FUNCTION ALGEBRAS

DEFINITION 4.1. A function class Φ on X is *complete* if γ_Φ is a complete Cauchy structure on Φ.

If ψ is a filter on Φ belonging to γ_Φ then for every $x \in X$ $\psi(\dot{x})$ converges in $\mathbb{R}$ and the function $\lim_p \psi$ defined as $\lim_p \psi(x) = \lim \psi(\dot{x})$ belongs to $\mathbb{R}^X$ and is the *pointwise limit of* ψ.

THEOREM 4.2. *A function class Φ is complete if and only if* $\lim_p \psi \in \Phi$ *whenever $\psi \in \gamma_\Phi$.*

Proof. Suppose Φ is γ_Φ-complete and let $\psi \in \gamma_\Phi$. There exists $f \in \Phi$ such that $\psi \cap \dot{f} \in \gamma_\Phi$. Then for every $x \in X$ the filter $(\psi \cap \dot{f})(\dot{x}) = \psi(\dot{x}) \cap \dot{f}(x)$ converges to $f(x)$. Hence $f = \lim_p \psi$. Conversely let $\psi \in \gamma_\Phi$ and suppose $f = \lim_p \psi$ belongs to Φ. Then since (Φ, γ_Φ) is a subspace of $(\Gamma(c(\Phi)), \widehat{c(\Phi)})$ (3.6) we have, using Theorem 1.3 of [5], that $\psi \cap \dot{f} \in \widehat{c(\Phi)}$. Since ψ and $\dot{f}$ both are filters on Φ, we have $\psi \cap \dot{f} \in \gamma_\Phi$. ∎

LEMMA 4.3. *If Φ is a function class on X, $(\hat{X}, \widehat{u(\Phi)})$ is the completion of X with the weak uniformity $u(\Phi)$ and $\hat{\Phi}$ is the collection of all extensions $\hat{f}$ of functions f belonging to Φ, then we have $\widehat{u(\Phi)} = u(\hat{\Phi})$.*

Proof. The maps in $\hat{\Phi}$ are all $\widehat{u(\Phi)}$-uniformly continuous so clearly $u(\hat{\Phi}) \leqslant \widehat{u(\Phi)}$. To prove the other inequality let U be a basic entourage in $u(\Phi)$ of the form $U = \{(x, y) \mid |f_i(x) - f_i(y)| < \varepsilon, i\colon 1 \cdots n\}$ where $f_1 \cdots f_n$ belong to Φ and $\varepsilon > 0$. If $\mathscr{M}$ and $\mathscr{N}$ are $u(\Phi)$-minimal Cauchy filters and $|\hat{f}_i(\mathscr{M}) - \hat{f}_i(\mathscr{N})| < \varepsilon/2$ for $i\colon 1 \cdots n$, then $]a_i - \varepsilon/2, a_i + \varepsilon/2[\in f_i(\mathscr{M}) \cap f_i(\mathscr{N})$ where a_i is the limit of $f_i(\mathscr{M})$. Choose $A_i \in \mathscr{M}, f_i(A_i) \subset]a_i - \varepsilon/2, a_i + \varepsilon/2[$ and $B_i \in \mathscr{N}, f_i(B_i) \subset]a_i - \varepsilon/2, a_i + \varepsilon/2[$ for $i\colon 1 \cdots n$. If we put $A = \bigcap_{i=1}^n A_i$ and $B = \bigcap_{i=1}^n B_i$ then $A \cup B \in \mathscr{M} \cap \mathscr{N}$ and $A \cup B$ is U-small. It follows that $\{(\mathscr{M}, \mathscr{N}) \mid |\hat{f}_i(\mathscr{M}) - \hat{f}_i(\mathscr{N})| < \varepsilon/2, \mathrm{i}\colon 1 \cdots n\} \subset \hat{U}$. ∎

THEOREM 4.4. *A function class Φ coincides with a collection $\Gamma(\mathscr{C})$ for some Cauchy structure $\mathscr{C}$ on X if and only if Φ is a complete function algebra containing the constants.*

Proof. If $\Phi = \Gamma(\mathscr{C})$ for some Cauchy structure $\mathscr{C}$ on X then clearly it is a function algebra containing the constants. From the results of Section 2 we have $\Phi = \Gamma(c(\Phi))$. So γ_Φ coincides with $\widehat{c(\Phi)}$ (3.6). From Theorem 1.3 in [5] it follows that (Φ, γ_Φ) is complete.

To prove the converse, suppose Φ is a γ_Φ-complete function algebra containing the constants. With the notations of Lemma 4.3 we have $\hat{\Phi} \subset \mathfrak{C}(\hat{\tau})$, where $\hat{\tau}$ is the topology on $\hat{X}$ of the completion of $u(\Phi)$. Since the extensions of the maps in Φ are unique, $\hat{\Phi}$ is a subalgebra of $\mathfrak{C}(\hat{\tau})$ containing the constants. Theorem 1.5 in [5] states that the map between $\Gamma(c(\Phi))$ and $\mathfrak{C}(\hat{\tau})$ sending f to its extension $\hat{f}$ is a homeomorphism when $\Gamma(c(\Phi))$ carries $\widehat{c(\Phi)}$ and $\mathfrak{C}(\hat{\tau})$ carries the continuous convergence structure. Since (Φ, γ_Φ) is a complete subspace of $(\Gamma(c(\Phi)), \widehat{c(\Phi)})$ it follows that $\hat{\Phi}$ is closed in $\mathfrak{C}(\hat{\tau})$ for the continuous convergence. Finally, $\hat{\Phi}$ is topology-generating since in view of Lemma 4.3 we have $\hat{\tau} = t(\hat{\Phi})$. From Theorem 58 in [1] we can conclude that $\hat{\Phi} = \mathfrak{C}(\hat{\tau})$. Consequently, taking the restrictions on X, we have $\Phi = \Gamma(c(\Phi))$. ∎

The following property answers problem (1) in [4].

COROLLARY 4.5. *A function class is composition closed if and only if it is a complete function algebra containing the constants.*

Proof. The result follows immediately from Theorem 4.4 and the characterization formulated in Section 2 that a function class Φ is composition closed if and only if $\Phi = \Gamma(c(\Phi))$. ∎

COROLLARY 4.6. *If $\mathscr{C}$ is a Cauchy structure on X and Φ is a subset of $\Gamma(\mathscr{C})$, then Φ coincides with $\Gamma(\mathscr{C})$ if and only if Φ is a complete subalgebra of $\Gamma(\mathscr{C})$ containing the constants such that $c(\Phi) = \mathscr{C}_u$.*

5. BOUNDED FUNCTIONS

LEMMA 5.1. *If Φ is a function class of bounded functions then $c(\Phi)$ is totally bounded. If $\mathscr{C}$ is a totally bounded Cauchy structure then $\Gamma(\mathscr{C})$ consists of bounded functions.*

Proof. If f is a bounded function and $\mathscr{W}$ is an ultrafilter on X then $f(\mathscr{W})$ converges. It follows that $c(\Phi)$ contains all ultrafilters when Φ consists of bounded functions.

Conversely, if $\mathscr{C}$ is totally bounded and $f \in \Gamma(\mathscr{C})$ then for every ultrafilter $\mathscr{W}$ on X we can choose a compact subset K of $\mathbb{R}$ and $W \in \mathscr{W}$ such that $f(W) \subset K$. We can select a finite number of ultrafilters $\mathscr{W}_i$, compact sets K_i and sets $W_i \in \mathscr{W}_i$, $f(W_i) \subset K_i$ for i: $1 \cdots n$, such that $W_1 \cup \cdots \cup W_n = X$. It follows that $f(X) = f(\bigcup_{i=1}^n W_i) = \bigcup_{i=1}^n f(W_i) \subset \bigcup_{i=1}^n K_i$. So $f(X)$ is bounded.

THEOREM 5.2. *A function class Φ consists of bounded functions if and only if $c(\Phi)$ is totally bounded. For a Cauchy structure $\mathscr{C}$ the collection $\Gamma(\mathscr{C})$ consists of bounded functions if and only if $\mathscr{C}_u$ is totally bounded.*

Proof. If $c(\Phi)$ is totally bounded then Φ consists of bounded functions since it is a subset of $\Gamma(c(\Phi))$. If $\mathscr{C}_u$ is totally bounded then $\Gamma(\mathscr{C})$ consists of bounded functions since $\Gamma(\mathscr{C}) = \Gamma(\mathscr{C}_u)$. The other implications are shown in the previous lemma. ∎

Remark 5.3. If $\Gamma(\mathscr{C})$ consists of bounded functions, $\mathscr{C}$ need not be totally bounded. For example, let $X = [0, 1]$, let $\mathscr{V}(x)$ be the usual neighborhoodfilter at x, fix a non-principal ultrafilter $\mathscr{U}_0 \supset \mathscr{V}(0)$ and put $\mathscr{C} = \{\mathscr{F} \mid \mathscr{F} \supset \mathscr{V}(x)$ for some $x \neq 0\} \cup \{\mathscr{F} \mid \mathscr{F} \supset \mathscr{V}(0), \mathscr{U}_0 \not\supset \mathscr{F}\}$. Then $\Gamma(\mathscr{C})$ coincides with the collection of continuous functions on X with the usual topology and hence it consists of bounded functions. However, $\mathscr{C}$ is not totally bounded.

THEOREM 5.4. *If Φ is a function class of bounded functions then γ_Φ coincides with the collection of Cauchy filters for the uniformity of uniform convergence.*

Proof. If Φ consists of bounded functions, $c(\Phi)$ is totally bounded so with the notations of 4.4, $(\hat{X}, \hat{\tau})$ is a compact space. So the structures of continuous convergence and of uniform convergence on $\mathfrak{C}(\hat{\tau})$ coincide. The continuous convergence on $\mathfrak{C}(\hat{\tau})$ is homeomorphic with the complete structure $\widehat{c(\Phi)}$ on $\Gamma(c(\Phi))$ and the uniform convergence on $\mathfrak{C}(\hat{\tau})$ is homeomorphic to the complete structure of uniform convergence on $\Gamma(c(\Phi))$. It follows that the induced Cauchy structures on Φ coincide. ∎

For the next two theorems we will give proofs depending on the previous sections. However, the results also follow from some theorems in [6].

THEOREM 5.5. *If Φ is a function class of bounded functions the following properties are equivalent.*

(1) *$\Phi = \Gamma(\mathscr{C})$ for some Cauchy structure $\mathscr{C}$ on X,*

(2) *$\Phi = U(\mathscr{U})$ for some (totally bounded) uniformity $\mathscr{U}$ on X,*

(3) *Φ is composition closed,*

(4) *Φ is a uniformly closed function algebra containing the constants.*

Proof. (1) $\Rightarrow$ (2) If $\Phi = \Gamma(\mathscr{C})$ let $u(\Phi)$ be the weak uniformity. Then $u(\Phi)$ is totally bounded and $\Phi \subset U(u(\Phi)) \subset \Gamma(c(\Phi)) = \Phi$.

(2) $\Rightarrow$ (3) If $\Phi = U(\mathscr{U})$ for some uniformity $\mathscr{U}$ on X, let $u(\Phi)$ again be the weak uniformity. $u(\Phi)$ is totally bounded and $\Phi = U(u(\Phi)) = \Gamma(c(\Phi))$.

(3) $\Rightarrow$ (4) (4.5) and (5.4).

(4) $\Rightarrow$ (1) (5.4) and (4.4).

Remark 5.6. For a uniformly closed function algebra Φ of bounded functions containing the constants the difference whether $\Phi = \Gamma(c(\Phi))$ or $\Phi = \mathfrak{C}^*(t(\Phi))$ lies in the separation: $\Phi = \Gamma(c(\Phi))$ holds if Φ is point separating, $\Phi = \mathfrak{C}^*(t(\Phi))$ holds if Φ separates the zerosets of $(X, t(\Phi))$ [13].

THEOREM 5.7. *If Φ is a function class of bounded functions then the algebra generated by $\Phi \cup$ {constants} is uniformly dense in $\Gamma(c(\Phi))$.*

Proof. The algebra $\langle \Phi \rangle$ generated by $\Phi \cup$ {constants} again consists of bounded functions and so does its uniform closure $\overline{\langle \Phi \rangle}$ in $\Gamma(c(\Phi))$. From (3.3) we have that $\overline{\langle \Phi \rangle}$ is again a function algebra. Applying 5.5 we can conclude that $\overline{\langle \Phi \rangle} = \Gamma(c(\Phi))$.

If Φ is a function class, let Φ^* be the collection of bounded functions in Φ. The collection Φ^* is in general not point separating even if Φ is point separating. For instance, if Φ is the collection of real polynomials on $\mathbb{R}$ then Φ^* only contains the constants. If Φ is an algebra containing the constants and closed under bounded inversion (i.e., $f \geqslant 1$ in Φ implies $1/f$ in Φ) then the point separability of Φ implies the point separability of Φ^* [6]. For instance, any collection $\Gamma(\mathscr{C})$ satisfies these conditions. So any collection $\Gamma^*(\mathscr{C})$ is point separating.

The proof of the following lemma is straightforward.

LEMMA 5.8. *If $\mathscr{C}$ is a Cauchy structure on X then so is $\mathscr{C}_t = \mathscr{C} \cup \{\text{ultrafilters on } X\}$. $\mathscr{C}_t$ is the finest totally bounded Cauchy structure on X coarser than $\mathscr{C}$. It is a reflection of $\mathscr{C}$ in the category of totally bounded Cauchy structures.*

THEOREM 5.9. *For every Cauchy structure $\mathscr{C}$ on X we have $\Gamma^*(\mathscr{C}) = \Gamma(\mathscr{C}_t)$.*

Proof. Since $\mathscr{C}_t \leqslant \mathscr{C}$ and $\mathscr{C}_t$ is totally bounded we clearly have $\Gamma^*(\mathscr{C}) \supset \Gamma(\mathscr{C}_t)$. If $f \in \Gamma^*(\mathscr{C})$ and $\mathscr{F} \in \mathscr{C}_t$ then either we have $\mathscr{F} \in \mathscr{C}$ or we have $\mathscr{F}$ is an ultrafilter and $f(\mathscr{F})$ is an ultrafilter on the bounded set $f(X)$. So in each of the two cases $f(\mathscr{F})$ converges. ∎

COROLLARY 5.10. *If Φ is composition closed then so is Φ^* and $c(\Phi^*) = (c(\Phi) \cup \{\text{ultrafilters on } X\})_u$.*

Proof. If Φ is composition closed then $\Phi = \Gamma(c(\Phi))$ and $\Phi^* = \Gamma((c(\Phi))_t)$. So Φ^* is composition closed (5.5) and we have $\Phi^* = \Gamma(c(\Phi^*)) = \Gamma((c(\Phi))_t)$. Hence $c(\Phi^*) = ((c(\Phi))_t)_u$. ∎

COROLLARY 5.11. *If $\mathscr{C}$ is a Cauchy structure on X and X has non-measurable cardinality then $c(\Gamma^*(\mathscr{C}))$ is the finest totally bounded uniformizable Cauchy structure on X coarser than $\mathscr{C}$. It is a reflection of $\mathscr{C}$ in the category of totally bounded uniformizable Cauchy spaces.*

THEOREM 5.12. *If Φ is a function lattice containing the constants then Φ^* is a (sequentially) dense subset of (Φ, γ_Φ).*

Proof. For $f \in \Phi$ and $n \in \mathbb{N}$ choose $f_n = f \vee (-n) \wedge n$ in Φ^* and let ψ be the Fréchet filter of the sequence $(f_n)_{n \geqslant 1}$. If $\mathscr{F}$ is a Φ-filter and $f(\mathscr{F})$ converges to $a \in \mathbb{R}$ then $\psi(\mathscr{F})$ also converges to a. So $\psi \cap \dot{f}$ belongs to γ_Φ and ψ converges to f. ∎

Remark 5.13. Φ^* is in general not a subspace of Φ. In fact the previous result and Theorems 4.4 and 5.10 imply that whenever Φ is a composition closed function class and $(\Phi^*, \gamma_{\Phi^*})$ is a subspace of (Φ, γ_Φ), then Φ and Φ^* coincide.

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